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A DYNAMIC CONTACT PROBLEM FOR AN ELASTIC HALF-PLANE IN THE CASE OF HIGH FREQUENCY OSCILLATIONS*

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Harmonic high frequency oscillations of a rigid stamp coupled without friction to an elastic half-plane are considered. The main difficulty in constructing the high-frequency asymptotic forms is that of carrying out the effective factorization of the kernel of the basic integral equation. A function is proposed, which takes into account all properties of the kernel, enables it to be uniformly approximated and is easily factorized. Such a solution of the problem of approximate factorization makes it possible to write, in a simple explicit form, the principal term of the asymptotic expression of the solution. The nature of the distribution of contact stresses under the stamp is studied, as well as the compliance of the foundation and phase shift between the applied force and the displacement of the stamp.

The problem was studied earlier in /1–4/ for the low-frequency case. Three classes of solutions were constructed in /5/, the low frequency solution, one effective at medium frequencies, and a high-frequency solution. The high frequency solution of /5/ however does not capture the true root-type singularities of the contact stress near the sharp edges of the stamp.

1. As we know /4, 5/, the problem in question can be reduced to the following integral equation:

$$\int_{-1}^{1} \varphi(t) K\left(\frac{x-t}{\lambda}\right) dt = \frac{G}{a} W, \quad |x| \leqslant 1 \tag{1.1}$$

$$K(x) = \frac{1}{2\pi} \int_{\sigma} L_*(u) e^{-iux}\, du, \quad \lambda^2 = \frac{G}{\rho \varkappa^2 a^2}, \quad \beta^2 = \frac{1-2\nu}{2(1-\nu)}$$

$$L_*(u) = \frac{\sqrt{u^2-\beta^2}}{4u^2\sqrt{u^2-\beta^2}\sqrt{u^2-1}-(2u^2-1)^2}$$

The dependence of all quantities on time is assumed to be of the type $f(x, t) = \operatorname{Re}[f(x) e^{-i\varkappa t}]$. In (1.1) $\varphi(x)$ is the contact stress amplitude, W is the stamp oscillation amplitude, λ is a parameter which is small at high frequencies, G, ν are the elastic constants, a is the stamp half-width and $\varkappa$ is the frequency of the oscillations. The initial Eq.(1.1) is equivalent to the following two Eqs./6/:

$$\int_{0}^{\infty} \omega(t) K(x-t)\, dt = \frac{1}{\lambda} + \int_{0}^{\infty} \left[\omega\left(\frac{2}{\lambda}+\tau\right) - r(\tau)\right] K(x+\tau)\, d\tau \tag{1.2}$$

$$\int_{-\infty}^{\infty} r(t) K\left(\frac{x-t}{\lambda}\right) dt = 1 \tag{1.3}$$

provided that

$$\varphi(x) = \frac{G}{a} W \left[\omega\left(\frac{1-x}{\lambda}\right) - \omega\left(\frac{1-x}{\lambda}\right) - r(x)\right] \tag{1.4}$$

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The problem in question represents a typical problem with singular perturbations in the small parameter λ.

In constructing the principal term of the asymptotic expression of the solution as $\lambda \to 0$, the last integral in (1.2) can often be neglected, and (1.2) then becomes the Wiener-Hopf equation on a semi-axis /6, 7/. This usually produces an error of order $\exp(-\varepsilon/\lambda)$ $(\varepsilon > 0)$ uniformly in x. In some instances the error is of the power-type, i.e. it becomes more singificant. Below we shall show that in the present problem the discrepancy caused by neglecting the term in question is small and of power type. Namely, we have the estimate

$$\int_0^\infty \left[\omega\left(\frac{2}{\lambda} + \tau\right) - v(\tau)\right] K(x + \tau)\, d\tau = O(\lambda^{1/2}), \quad \lambda \to 0 \tag{1.5}$$

uniformly in x.

In what follows, our aim will be to construct the principal term of the asymptotic form of the solution. The estimate (1.5) leads to the conclusion that the principal term of (1.4) is found by solving the problems for two semi-infinite stamps (1.2) and one infinite stamp (1.3). Physically, this means that the high-frequency oscillations are of such short wavelength, and that the perturbations occurring at the right end of the stamp have, in general, practically no influence on the wave processes at the left end, and vice versa.

The solution of the equation on the axis (1.3) is constructed using a Fourier transformation and has the form

$$v(x) = -i(\beta\lambda)^{-1} \equiv v \tag{1.6}$$

To obtain, successfully, a solution of the equation on the semi-axis

$$\int_0^\infty \omega(t) K(x - t)\, dt = \frac{1}{\lambda} \tag{1.7}$$

we must factorize the kernel symbol. Since the solution of (1.7) is stable under small perturbations of the symbol on the real axis /4/, we shall factorize it approximately.

2. The kernel symbol $L_*(u)$ represents a combination of four radicals $\sqrt{u+\beta}, \sqrt{u+1}, \sqrt{u-\beta}, \sqrt{u-1}$ with branch points. Let us produce, in the plane of the complex variable u, the cuts connecting the points $-\beta$ and -1 with infinity in the half-plane, and points β and 1 with infinity in the upper half-plane. In addition to the branch points on the real axis, the symbol has two Rayleigh poles $u = \pm u_1$, $u_1 > 1$. According to the principle of limit absorption the contour σ in (1.1) coincides with the real axis passing the positive singularities from below and the negative ones from above.

The function $L_*(u)$ shows a qualitatively different behaviour on different segments of the real axis. When $|u| \geqslant 1$ the function is real, it becomes complex when $\beta < |u| < 1$, and imaginary when $|u| \leqslant \beta$.

Let us approximate the symbol $L_*(u)$ by the expression

$$L_* = \sqrt{u^2 - \beta^2}\, \frac{4u^2\sqrt{u^2-\beta^2}\sqrt{u^2-1} + (2u^2-1)^2}{16u^4(u^2-\beta^2)(u^2-1) - (2u^2-1)^4} \approx \tag{2.1}$$

$$\frac{A\sqrt{u^2-\beta^2}}{(u^2-u_1^2)(u^2-z^2)(u^2-\bar{z}^2)} M_+(u) M_-(u) = L(u)$$

$$\operatorname{Im} z > 0, \quad A > 0, \quad B > 0$$

$$M_\pm(u) = Bu\sqrt{u \pm \beta}\sqrt{u \pm 1} - (\sqrt{2}\, u \pm 1)^2$$

The function $L(u)$, as well as $L_*(u)$, is even, has two Rayleigh poles $u = \pm u_1$ and exhibits the same qualitative behaviour on different segments of the real axis. In addition, it captures the behaviour exactly only at zero and infinity. It also has the true sign of the imaginary part, which is important when satisfying the uniqueness theorem /4/. The expression $M_+(u)$ has a zero in the upper half-plane, and the zero must be cancelled by the zero in the denominator $u = -\bar{z}$. We note that the point $u = -z$ represents its second zero.

Taking all that has been said into account, we can factorize the function $L(u)$ thus

$$L_+(u) = \frac{A\sqrt{u+\beta}}{(u+u_1)(u+z)(u+\bar{z})} M_+(u) \tag{2.2}$$

3. Applying the Wiener-Hopf method to Eq. (1.7) and taking the factorization (2.2) into account, we arrive at the following expression for the Fourier transform of the function $\omega(x)$:

$$\Omega_+(u) = \frac{C}{u}\frac{(u+u_1)(u+z)(u+\bar{z})}{\sqrt{u+\beta}\, M_+(u)} = \frac{C}{u}\frac{(u+u_1)(u+z)(u+\bar{z})}{\sqrt{u+\beta}\, \Delta(u)} \times \tag{3.1}$$

$$[Bu\sqrt{u+\beta}\sqrt{u+1} - (\sqrt{2}\, u + 1)^2], \quad C = \frac{u_1 |z|^2}{A\lambda\sqrt{\beta}}$$

$$\Delta(u) = B^2u^2(u+\beta)(u+1) - (\sqrt{2}u + 1)^4 = b(u+z)(u+\bar{z})(u+\eta)(u+\bar{\eta}), \quad b = B^2 - 4 \tag{3.2}$$

Thus we have

$$\Omega_+(u) = \frac{C(u+u_1)}{bu(u+\eta)(u+\bar\eta)\sqrt{u+\beta}}\left[Bu\sqrt{u+\beta}\sqrt{u+1} - (\sqrt{2}u+1)^2\right] = \tag{3.3}$$
$$\frac{C}{b}\left[a_1\frac{\sqrt{u+1}}{u+\eta} + a_2\frac{\sqrt{u+1}}{u+\bar\eta} + \frac{a_3}{u\sqrt{u+\beta}} + \frac{a_4}{(u+\eta)\sqrt{u+\beta}} + \frac{a_5}{(u+\bar\eta)\sqrt{u+\beta}}\right]$$
$$a_1 = B\frac{u_1-\eta}{\bar\eta-\eta}, \quad a_2 = \bar a_1, \quad a_3 = -\frac{u_1}{|\eta|^2}, \quad a_4 = \frac{(u_1-\eta)(\sqrt{2}\eta-1)^2}{\eta(\bar\eta-\eta)}$$
$$a_5 = \bar a_4$$

This makes it possible to write the function $\omega(x)$ in explicit form, since we have the following inversion formulas:

$$\frac{1}{(u+\eta)\sqrt{u+\beta}} \Leftarrow -i\frac{e^{i\eta x}}{\sqrt{\beta-\eta}}\operatorname{erf}\sqrt{-ix(\beta-\eta)} \tag{3.4}$$
$$\frac{\sqrt{u+1}}{u+\eta} \Leftarrow \frac{e^{-i(\pi/4-x)}}{\sqrt{\pi x}} - ie^{i\eta x}\sqrt{1-\eta}\operatorname{erf}\sqrt{-ix(1-\eta)}, \quad x>0$$

The explicit form of the function $\omega(x)$ as well as Eq.(3.1) yield directly the following estimate:

$$\omega(x) \sim v + ae^{i\beta x}x^{-3/2}, \quad x\to+\infty \tag{3.5}$$

which shows that the boundary-layer solution, as we expected, transforms in the outer zone into the solution of the problem on an infinite stamp (1.6). The process however is very slow.

Adding the estimate (3.5) and the obvious estimate

$$K(x) \sim c_1e^{iu_1x} + c_2e^{i\beta x}x^{-3/2} + c_3e^{ix}x^{-3/2}, \quad x\to+\infty \tag{3.6}$$

we can confirm the validity of relation (1.5).

Let us find the relation connecting the force acting on the stamp with its subsidence (assuming that the stamp is weightless)

$$P = a\int_{-1}^{1} q(x)\,dx = GW\int_{-1}^{1}\left[\omega\left(\frac{1+x}{\lambda}\right) + \omega\left(\frac{1-x}{\lambda}\right) - v\right]dx = \tag{3.7}$$
$$GW\left\{2v + 2\int_{-1}^{\infty}\left[\omega\left(\frac{1+x}{\lambda}\right) - v\right]dx\right\}$$

The latter equality holds by virtue of estimate (3.5), with an error of order $O(\lambda^{3/2})$ equal to the error in the solution of (1.1). Further, since

$$\int_{-1}^{\infty}\left[\omega\left(\frac{1+x}{\lambda}\right) - v\right]dx = \lambda\int_{-\infty}^{\infty}[\omega(t) - vH(t)]\,dt = \lambda\,[\Omega_+(u) - V_+(u)]_{u=0} \tag{3.8}$$

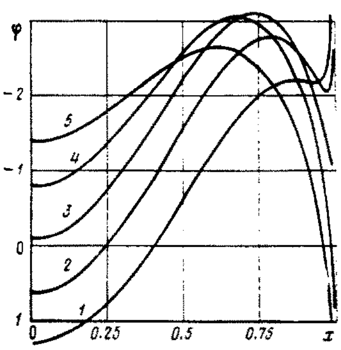

Fig.1

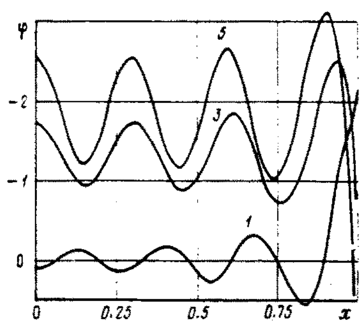

Fig.2

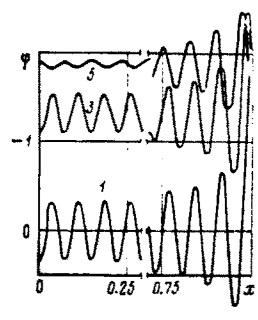

Fig.3

($H(t)$ is the Heaviside function) we finally obtain the following expression for the compliance of the foundation:

$$W = \frac{P}{2G}\,\frac{\lambda}{-i\beta^{-1}+\lambda CD} \qquad D = \frac{1}{b}\left\{2\operatorname{Re}\left[\frac{1}{\eta}\left(a_1 + \frac{a_4}{\sqrt{\beta}}\right)\right] - \frac{a_3}{2\beta^{3/2}}\right\} \tag{3.9}$$

We note that the constants C and D in (3.9) are real; therefore the phase shift between the depression and applied force is given by the formula

$$\theta = -\operatorname{arctg}(\beta CD\lambda)^{-1} \tag{3.10}$$

The formula for the amplitude of the subsidence is also easily obtained.

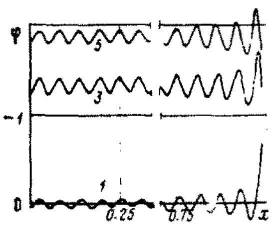

Fig.4

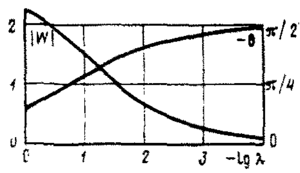

Fig.5

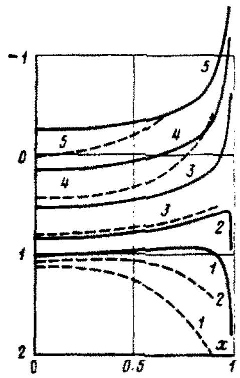

Fig.6

It is clear that the solution of the problem for a weightless stamp can be converted to that for a stamp of any mass. We achieve this by multiplying it by a complex factor.

4. All computations were carried out for $\nu = 0.3$. In this case we have $u_1 = 1.07827$. The values $A = 0.075330$; $B = 1.04835$; $z = 0.50252 + 0.044849\,i$ of the coefficients obtained ensure the uniform approximation (2.1) with exact values at zero and infinity, and with a maximum relative error of less than 10%. At the most "critical" point, i.e. at the Rayleigh pole $u = u_1$ the error does not exceed 2%, and at the branch points $u = \beta$, $u = 1$ it does not exceed 9%. The value of the constant in (3.2) is $\eta = 1.15679 + 0.12688i$.

The contact stress was calculated using the explicit formulas (1.4), (3.4). Figs.1–4 show the curves for the contact stresses referred to $GW_1(a\lambda)$ respectively, for the values of λ equal to $1/5, 1/40, 1/160, 1/2560$. Curves 1–5 correspond to the times $t = 0, \pi/8, \pi/4, 3\pi/8, \pi/2$ taken in the quarter period $0 \leqslant t \leqslant \pi/2$. The qualitative change in the form of the distribution of the contact stresses under the stamp is clearly seen. For medium values of λ ($1/5$) the graph is smooth. At moderately small λ ($1/40$ and $1/160$) the oscillations in the outer zone are superimposed on the fundamental (constant) value (see (3.5)) and for these values λ they are still real. For very small λ ($1/2560$) the effect of the oscillation is very insignificant in the outer zone, and is retained only in a narrow boundary layer.

Fig.5 shows the dependence of the amplitude modulus $|W|$ referred to $P/(2G)$ and phase shift (3.10) on λ. We note that the degenerate solution (1.6) would give a constant value of $\theta = -\pi/2$ for the phase shift.

Fig.6 illustrates the attempt to compare the computed expressions $\pi a q(x)/P$ for $m_0 = 0.4$ in the framework of the proposed method (solid lines) for large values of $\lambda = 1$, with the solution given in /3/, p.74 (dashed lines) obtained with the help of the method of orthogonal polynomials in which 2–3 terms were taken. Here it was taken into account that the time dependence in /3/ was apparently taken in the form $f(x, t) = \operatorname{Im}[f(x)e^{i\omega t}]$. We discover the qualitative agreement of the form of behaviour of the contact stress curves. This implies that the method proposed here can be used for medium values of λ of the order of several tens.

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